Grid spanners with low forwarding index for energy efficient networks ⋆

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Abstract

A routing R of a connected graph G is a collection that contains simple paths connecting every ordered pair of vertices in G. The *edge-forwarding index with respect to* R (or simply the forwarding index with respect to R) $\pi(G, R)$ of G is the maximum number of paths in R passing through any edge of G. The *forwarding index* $\pi(G)$ of G is the minimum $\pi(G, R)$ over all routings R's of G. This parameter has been studied for different graph classes [12], [1], [5], [4]. Motivated by energy efficiency, we look, for different numbers of edges, at the best spanning graphs of a square grid, namely those with a low forwarding index.

Keywords: spanning subgraphs, forwarding index, energy saving, routing, grid.

1 Introduction

A routing R of a given connected graph G of order N is a collection of $N(N-1)$ simple paths connecting every ordered pair of vertices of G. The routing R induces on every edge e a *load* that is the number of paths going through e. The *edge-forwarding index* (or simply the forwarding index) $\pi(G, R)$ of G with respect to

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R is the maximum number of paths in R passing through any edge of G. It corresponds to the maximum load over all edges of the graph when R is used. Therefore, it is important to find routings minimizing this index. The forwarding index $\pi(G)$ of G is the minimum $\pi(G, R)$ over all routings R's of G. This parameter has been studied for different graph classes (examples can be found in [1], [5], [4]) and this survey [12] gives a global view on the known results.

We call a connected spanning subgraph of a graph G, a *spanner* of G. More precisely, it is a connected subgraph that has the same set of vertices as G. Our goal is to find, for a given bound on the number of edges, the best spanner of G, namely the one with the minimum forwarding index. The problem can also be viewed as: for a given bound U on the forwarding index, find a spanner F of G with minimum number of edges such that $\pi(F) \leq U$.

Knowing how to solve this problem is very interesting in practice for network operators willing to reduce the energy consumed by their networks. In fact, most of the network links consume a constant energy independently of the amount of traffic they are flowing [2], [11]. Therefore, it was proposed to reduce the energy used by the network links by turning some of them off, or more conveniently, putting them into an idle mode. Outside the rush hours, several studies [3], [10], [6], [7], [8], show that a good choice of the links to turn off can lead to significant energy savings, while keeping the same communication quality. In the case where the throughputs from every node to every other node are of the same order, and where the capacities also lie in the same small range, a good choice of those links is reduced to the problem of finding spanners of the network with low forwarding indices.

In this paper, we consider the case in which the initial graph is a square grid. We consider the asymptotic case with n large. We have two main contributions.

On one side, we know that the forwarding index of the $n \times n$ grid G_n is $\frac{n^3}{2}$, see Proposition 1.1 [6]. G_n has $2(n-1)^2 \sim 2n^2$ edges. An important remark is that the load of the edges is lower in the corner than in the middle of the grid. Using that, we show that we can build spanners of G_n with much fewer edges (only $13/18 \approx 72\%$ of the edges) and the same forwarding indices as G_n. We show that the proposed spanners are close to optimum in the sense that we prove that it is impossible to build spanners with fewer than $4/3n^2$ edges (66% of the edges).

On the other side, the smallest possible spanner of the $n \times n$ grid G_n is a spanning tree. The forwarding index of the best spanning tree is asymptotically $\frac{3n^4}{8}$, see Proposition 1.2 [6]. When we add edges and consider spanners with a larger number of edges, the load on the edges decreases, and so does the forwarding index. In this paper, we study how the forwarding index decreases, when we increase the number of edges. The following table summarizes our results:

	Spanning tree	Spanners For an integer a, $2 \leq a \leq n$		Grid
forwarding index	$\frac{3}{8}n^4$	$\frac{1}{2a}n^4$	$\frac{1}{2}n^3$	$\frac{1}{2}n^3$
lower bound on number of edges	$n^2 - 1$	$\simeq n^2 + \frac{4}{9}(0.1a)^2$	$\frac{12}{9}n^2$	
number of edges in constructions	$n^2 - 1$	$n^2 + \frac{4}{9}a^2$	$\frac{13}{9}n^2$	$2n^2$

Proposition 1.1 *[6] The forwarding index of G_n is asymptotically $\frac{n^3}{2}$.*

Proposition 1.2 *[6] For $n \geq 3$, The spanning tree of G_n with the minimum forwarding index is a tree with centroid of degree 4 and 4 branches of almost equal sizes. its forwarding index is asymptotically $\frac{3n^4}{8}$.*

Note that, due to lack of space, parts of the proofs are omitted. Full proofs can be found in [9].

2 Spanners with the forwarding index of the grid, $\frac{n^3}{2}$, but much fewer edges

In this section, we first show that a spanner with the forwarding index of the grid has at least $\frac{4n^2}{3} = \frac{12n^2}{9}$ edges. We then provide spanners with $\frac{13n^2}{9}$ edges. But, before, we present some notations that will be used throughout the paper.

Notations. We note by $G_n = (V_n, E_n)$ the $n \times n$ square grid, where V_n is the set of Vertices and E_n is the set of edges. A square grid can always be seen as n rows intersecting n columns. We name $v(r, c)$ the vertex at the intersection of row $r \in [n]$ with column $c \in [n]$, where $[n]$ denotes the interval of the integer numbers between 1 and n. An edge joining $v(r, c)$ to $v(r, c + 1)$ is named $e_h(r, c)$ and an edge joining $v(r, c)$ to $v(r + 1, c)$ is named $e_v(r, c)$.

Proposition 2.1 *For any F spanner of G_n such that $\pi(F) \leq \frac{n^3}{2}$, F must have, asymptotically, at least $\frac{4n^2}{3}$ edges.*

The proof of the proposition can be found in [9].

Theorem 2.2 *There exists F_n a spanner of G_n such that $\pi(F_n) \sim \frac{n^3}{2}$ and its number of edges is asympotically equal to $\frac{13n^2}{9}$.*

Proof. Let us first explain the intuition behind the construction of the spanner of the grid, F_n. We know from the proof of Proposition 2.1 the ratio of edges needed in

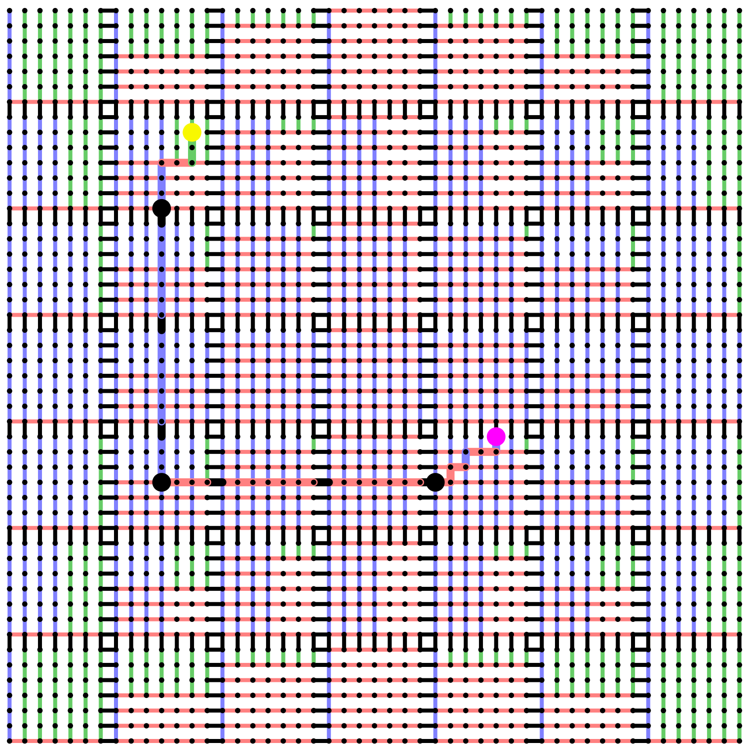

Fig. 1. Construction of the spanner F_n of Theorem 2.2, for $n = 7^2$, and an example of path of the routing R of F_n (from the yellow vertex to the pink vertex).

every row or column in order to satisfy the lower bound. We cut the grid into small squares. Then, according to the position of the square, we use only the number of needed horizontal edges and vertical edges in each square according to the lower bound. It turns out that adding only few edges to ensure the connectivity is enough to get a spanner F_n with a routing R such that $\pi(F_n, R) \sim \frac{n^3}{2}$.

Construction of F_n. Let k be an integer number such that $1 \leq k \leq n$. We divide G_n into small square grids of size $k \times k$. We do so by partitioning vertices of G_n into $(\frac{n}{k})^2$ sets $S_{(i,j)}$ with $i \in [\frac{n}{k}]$ and $j \in [\frac{n}{k}]$: $S_{(i,j)} = \{v(r,c) \in V_n; i-1 < \frac{r}{k} \leq i, j-1 < \frac{c}{k} \leq j\}$. We call a vertex in $S_{(i,j)}$ that has a neighbour in G_n outside $S_{(i,j)}$ a border vertex.

Let us now describe a spanner F_n that verifies our theorem. An example of it is shown in Figure 1 in the case of $n = k^2 = 7^2$. Let t be the function defined on integers by $t(x) = \lceil 4xk(n-xk)k/n^2 \rceil$. It represents the number of needed columns (respectively rows) for a square that is on the x-th position horizontally (respectively vertically). We build F_n starting from a subgraph that has all vertices of G_n and no edges. For every $S_{(i,j)}$, $i, j \in [\frac{n}{k}]$, we choose edges to connect vertices in $S_{(i,j)}$ in the following way:

- we add to F_n all edges $e_v(r,c)$ such that $(r \mod k) \in \{1,\ldots,t(i)\}$ (red edges in Figure 1) and
- all edges $e_h(r,c)$ such that $(c \mod k) \in \{1,\ldots,t(j)\}$ (blue edges in Figure 1);
- then we add to F_n simple paths just to connect the remaining independent vertices (green edges in Figure 1).
- We then add all edges that do not have both endpoints in the same set $S_{(i,j)}$ (black edges in 1). We show in the following that adding all of them is not strictly necessary.

Description of the routing R. We now give a routing of the spanner F_n, R. For every ordered pair of vertices $(v(r_a,c_a), v(r_b,c_b))$ of V_n, we describe the path connecting $v(r_a,c_a)$ to $v(r_b,c_b)$ in R. We distinguish two types of ordered pairs of vertices:

- Type-1 pairs: $\lceil r_a/k \rceil = \lceil r_b/k \rceil$ or $\lceil c_a/k \rceil = \lceil c_b/k \rceil$. Notice that this type includes ordered pairs with vertices that belongs to the same set $S_{(i,j)}$.
- Type-2 pairs: All the ordered pairs that do not belong to the first type.

For the Type-1 pairs, R uses the shortest path routing. For Type-2 pairs, R uses a three-segment path. An example of such path is shown in Figure 1. We name $i_a = \lceil r_a/k \rceil$, $i_b = \lceil r_b/k \rceil$, $j_a = \lceil c_a/k \rceil$ and $j_b = \lceil c_b/k \rceil$:

- Step-1: Let $i_m = \min(i_a, i_b, n/k - i_a, n/k - i_b)$ and $j_m = \min(j_a, j_b, n/k - j_a, n/k - j_b)$. The first segment is the shortest path from $v(r_a,c_a)$ to one of the two border vertices of $S_{(i_a,j_a)}$ that are on row $k(i_a-1)+t(j_m)$. Among the two vertices, we choose $v(r_x,c_x)$, which has the smallest distance to $S_{(i_a,j_b)}$ (as the first black vertex on the route in Figure 1).
- Step-2: Similarly, two border vertices of $S_{(i_b,j_b)}$ are on column $k(i_b-1)+t(i_m)$. Among these two vertices, $v(r_y,c_y)$ is the one that has the smallest distance to $S_{(i_a,j_b)}$ (as the third black vertex on the route in Figure 1). The second segment will be linking $v(r_x,c_x)$ to $v(r_y,c_y)$ by using the path $[v(r_x,c_x)v(r_x,c_y)v(r_y-c_y)]$, which is the shortest path from $v(r_x,c_x)$ to $v(r_x,c_y)$ composed of the two direct paths $[v(r_x,c_x)v(r_x,c_y)]$, following row r_x, and $[v(r_x,c_y)v(r_y-c_y)]$, following column c_y.
- Step-3: The third and last segment will be the shortest path from $v(r_y,c_y)$ to $v(r_b,c_b)$.

The computation of the number of edges of F_n and of its forwarding index can be found in [9]. $\square$

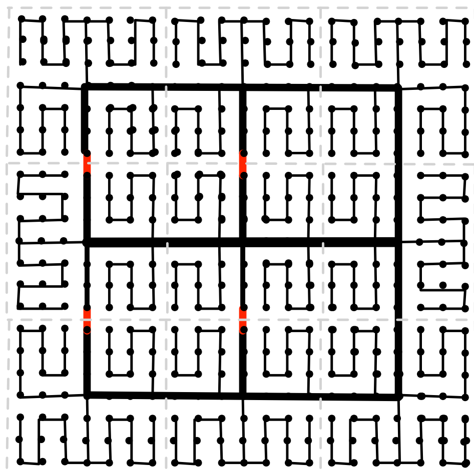

Fig. 2. Spanner of Proposition 3.1 for $n = 21$ and $a = 3$. Edges of the $a \times a$ grid are in bold. Edges that are not in a spanning tree of G_n are in red. Sectors with $(n/a)^2 = 7^2$ vertices are separated by dashed gray lines.

3 Spanners with forwarding indices in the range $]\frac{n^3}{2}, \frac{3n^4}{8}[$ and Lower bounds

We first provide spanners with forwarding indices in the range $]\frac{n^3}{2}, \frac{3n^4}{8}[$ in Proposition 3.1. We then prove that these spanners have a number of edges of the optimum order, see Proposition 3.2

3.1 Spanners' constructions

Proposition 3.1 *Let a be an integer such that, $2 \leq a \leq n$. There exists a spanner $F_n(a)$ of G_n with asymptotically $n^2 + \frac{4}{9}a^2$ edges and $\pi(F_n(a)) \leq \frac{n^4}{2a}$.*

Proof. We build a spanner of G_n, $F_n(a)$, in the following way. We divide the grid into a^2 sectors. A point is in Sector (i, j) if its coordonnates in the grid (x,y) are such that $\frac{n}{a}i \leq x < \frac{n}{a}(i+1)$ and $\frac{n}{a}j \leq y < \frac{n}{a}(j+1)$. Each of these sectors has $(n/a)^2$ vertices. We call *center* of the sector (i, j) the vertex $((i + 1/2)n/a, (j + 1/2)n/a)$. We consider the $a \times a$ subgrid linking all the sectors' centers. We then connect all the remaining vertices of a sector to its center with a spanning tree. This way, we get $F_n(a)$. Figure 2 provide a sketch of the construction of the spanner.

We now build a routing R for $F_n(a)$. The demand between two vertices of the same sector are routed on the tree spanning their sector using the unique shortest path between them. The demand between two vertices of different sector is first routed to their centers, and then is routed in the $a \times a$ grid.

The computation of the load of the routing R and of the number of edges of the spanner $F_n(a)$ can be found in [9]. □

3.2 Lower bounds

Proposition 3.2 *There exist no spanners of G_n with n^2+p^2 edges and a forwarding index less than $\frac{1}{9\sqrt{12}}\frac{n^4}{p} \simeq 0.032\frac{n^4}{p}$.*

The proof of the proposition can be found in [9].

4 Conclusion

We succeeded at providing spanners of the $n \times n$ grid with a small number of edges for a given forwarding index. Such spanners are important for energy efficient networks in which the traffic has to be routed in the network while using a minimum number of equipments. The unused equipments are then turned off to save energy. We leave as open two problems.

We propose spanners with a number of edges of optimum order for a forwarding index. More precisely, we have provided spanners of the grid with $n^2 + \frac{4}{9}a^2$ edges and forwarding indices $\frac{1}{2a}n^4 (2 \leq a < n)$. We proved that it is impossible to have spanners with the same FI and fewer than $\simeq n^2 + \frac{4}{9}(0.1a)^2$ edges. It would be very nice to succeed in filling the gap between the lower bounds and the constructions.

Similarly, we describe spanners with $13/9n^2$ edges and with the same forwarding index of the full grid G_n. We proved that spanners with such a forwarding index should have at least $12/9n^2$ edges. Would it be possible to find spanners with such a number of edges?

Last, we focused on a specific network in this work, the square grid. We are also interested by more general graphs. In particular, the arguments to derive lower bounds can be used for more general planar graphs with bounded degrees. It would be very interesting to find results and constructions for other families of planar graphs.

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